The asymptotic form of the laminar boundary-layer mass-transfer rate for large interfacial velocities

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The convective diffusion of matter from a stationary object to a moving fluid stream is distinct from pure heat transfer because of the appearance of a finite interfacial velocity at the solid surface. This velocity is related to the rate of mass transfer by a dimensionless group B in such a way that for $-1 < B < 0$ the transfer is from the bulk to the surface while for $0 < B < \infty$ the transfer is from the surface to the main stream. In this paper, asymptotic solutions to the two-dimensional laminar boundary-layer equations are developed for the case $B \gg 1$, and for rather general systems. It is shown that in most instances the asymptotic expressions for the rate of mass transfer become accurate when $B > 3$ and that the transition region between the pure heat-transfer analogy ($B \sim 0$) and the $B \gg 1$ asymptote may be described by a simple graphical interpolation. These results may easily be extended to three-dimensional surfaces of revolution by the usual co-ordinate transformations of boundary-layer theory.

1. Introduction and basic equations

It is well known by now that, although it is possible to consider heat and mass transfer as completely analogous phenomena under certain special conditions, there are important fundamental differences between these two processes which in general must be taken into account. It has been repeatedly demonstrated, for example, that in the rather common case of mass or heat exchange taking place separately between a stationary surface and a moving fluid, the rate of mass transfer may be appreciably different from the rate of heat exchange, even for systems with identical external flow configurations and transport parameters. This is so for the following reason: if the fluid is assumed to have constant properties, then the equations of motion remain uninfluenced by the heat transfer process. In the case of mass exchange, however, the surface plays the additional role of acting either as a source or as a sink of material, with the result that a net hydrodynamic velocity normal to the solid-fluid interface is thereby established. It is quite clear, therefore, that if this interfacial velocity is appreciable in magnitude it can cause a distortion in the velocity profile which would normally exist in the absence of mass exchange, and thus substantially affect the transfer rate of matter. One can conclude then that, even though an analogy could still exist under certain circumstances if the two processes were to take place simultaneously, pure heat transfer and mass transfer will in general not obey the same mathematical relations and will have therefore to be studied separately. It is the purpose of this article to investigate this point theoretically for laminar boundary-layer flows and to show under what conditions one can expect this interfacial velocity to play a significant role and to affect the often postulated analogy between pure heat transfer and mass exchange.

The usual two-dimensional laminar boundary-layer equations of continuity, momentum and mass transfer will now be used as our starting-point. We have, in conventional boundary-layer notation and for a constant-property fluid (Schlichting 1960), that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{Sc}\frac{\partial^2 \theta}{\partial y^2}, \tag{3}$$

where for simplicity both dimensionless and stretched laminar boundary-layer co-ordinates have been employed. In other words, if U_∞, L and ν are respectively the characteristic velocity, the characteristic length and the kinematic viscosity of the system, then

$$\begin{aligned} x &= (\text{distance along surface})/L, \\ y &= (U_\infty L/\nu)^{\frac{1}{2}}\,(\text{distance normal to surface})/L, \\ u &= (\text{velocity component along } x)/U_\infty, \\ v &= (U_\infty L/\nu)^{\frac{1}{2}}\,(\text{velocity component along } y)/U_\infty. \end{aligned}$$

Similarly, Sc is the standard Schmidt number ν/D, $U(x)$ is the dimensionless potential-flow distribution at the edge of the boundary layer, and θ is the dimensionless 'concentration' $(W-W_\infty)/(W_s-W_\infty)$, where W is the weight fraction of the diffusing species and where the subscripts s and ∞ refer, respectively, to the surface and to the bulk. The mathematical system is finally made determinate by specifying the appropriate boundary conditions (Eckert & Drake 1959):

$$u=U(x),\quad \theta=0 \quad \text{at} \quad y=\infty,\quad x=0; \qquad u=0,\quad \theta=1 \quad \text{at} \quad y=0; \tag{4}$$

and

$$v=-\frac{B}{Sc}\frac{\partial\theta}{\partial y} \quad \text{at} \quad y=0, \tag{5}$$

where

$$B\equiv (W_s-W_\infty)/(1-W_s). \tag{6}$$

It should be carefully noted at this point that the mathematical model is not entirely general, but that it must satisfy a number of restrictions. It is obvious first of all that the Reynolds number $(U_\infty L/\nu)$ must be sufficiently large for the boundary-layer equations to hold and that the fluid properties, density, viscosity and diffusion coefficient, must be independent of composition. It must also be kept in mind that the equations, as written, are applicable only to a binary mixture consisting of one transferring species and an inert substance and not to the more complicated case of multi-component mass transfer which we shall not consider at this time. And finally, it should be realized that the analysis will be limited to those cases where the interfacial velocity, although large enough to affect the velocity profile inside the boundary layer, will not be so large as to cause a change in the pressure distribution around the surface, so that the function $U(x)$ appearing in equation (2) will still be given by the inviscid flow theory in the absence of mass exchange. As can easily be shown, this assumption is permissible as long as the Reynolds number is sufficiently large (Acrivos 1960*a*).

It is appropriate now to consider briefly equation (5) which expresses the normal velocity component at the solid-fluid surface in terms of the rate of transfer of the diffusing species. It has been repeatedly established in the past that equation (5) can be derived in a straightforward manner (Spalding 1954; Eckert & Drake 1959) from the requirement that the net rate of inert transfer into the surface be zero, since the solid interface is assumed to act either as a source or a sink for the diffusing species alone. In addition, as has already been pointed out by Spalding (1960) and Merk (1959*b*), this interfacial velocity is intimately connected to the dimensionless group B, defined by equation (6), which in a sense provides us with a qualitative measure of the relative importance of the coupling between momentum and mass transfer. In other words, if $B \sim 0$, then the system of equations would become indentical with that describing pure energy transfer in the absence of mass exchange, except that the Schmidt number in equation (3) would have to be replaced by the Prandtl number. This would mean then that only if $B \sim 0$ would one expect the analogy between pure heat transfer and mass exchange to hold, and that even for moderate values of this parameter one would have to consider these two operations as fundamentally distinct.

In the general case then, $B \neq 0$ with $-1 < B < 0$ if the exchange process is from the bulk to the surface, and $0 < B < \infty$ if the transfer is from the surface to the main stream. This in turn not only complicates one of the boundary conditions in the manner already explained but it does introduce a modification in the expression for the mass flux of the diffusing species at the surface. In other words, if the symbol j is used to denote the rate of transfer at the solid interface of the diffusing substance, in g/sec cm^2, then in view of the definition of v and y

$$j = \rho \left[U_\infty \frac{v}{(U_\infty L/\nu)^{\frac{1}{2}}} W_s - \frac{D}{L}\left(\frac{U_\infty L}{\nu}\right)^{\frac{1}{2}} (W_s - W_\infty) \frac{\partial \theta}{\partial y} \right]_{y=0},$$

which, because of equation (6), may be simplified into

$$\frac{jL}{\rho D}\left(\frac{\nu}{U_\infty L}\right)^{\frac{1}{2}} = -B\left(\frac{\partial \theta}{\partial y}\right)_{y=0}. \tag{7}$$

It is clear now that in order to evaluate the rate of mass transfer one is first required to determine the quantity $(\partial\theta/\partial y)_{y=0}$, identically equal to the analogous term in pure heat transfer only if $B \to 0$ and $Sc = Pr$, which as can be seen from equations (1) to (6) is a function of x, Sc, B and the geometry of the surface which fixes $U(x)$. This cannot in general be accomplished analytically, and it is for this reason that the mathematical problem has in the past been attacked from two different points of view. One method (Livingood & Donoughe 1955; Eckert & Hartnett 1957; Stewart & Prober 1961) has dealt with the numerical solution of the boundary-layer equations for the special class of wedge-like surfaces, for which it is possible to use the standard similarity transformations of boundary-layer theory to reduce the system to ordinary differential equations. This approach is without doubt of considerable value, but with rather obvious limitations on account of the presence of three independent parameters in the equations: the wedge angle, the Schmidt number and B. Alternatively, it has been suggested (Eckert & Lieblein 1949; Spalding 1954, 1961; Spalding & Evans 1961) that the problem could be handled by various extensions of the well known approximate von Karman–Pohlhausen integral technique, but, in view of Merk's (1959*b*) remarks, one might expect such methods to be both cumbersome and, under certain conditions, relatively inaccurate.

It is believed therefore that a more promising point of view would be to investigate the solution of the boundary-layer equations in the asymptotic extreme of large interfacial velocities where the breakdown of the analogy between pure heat transfer and mass exchange would be especially pronounced. In this way, first of all, the analysis would be carried out in the region of most interest where the effects of the interfacial velocity would be particularly significant. Secondly, if, as one would hopefully expect, it were possible to derive a closed-form expression for the rate of mass transfer in this asymptotic limit of large interfacial velocities, then, judging from past experience, one might also be able to describe with reasonable accuracy the interval intermediate between the pure heat transfer result and the asymptotic one by means of a simple graphical interpolation. But a final and hardly a negligible advantage of the asymptotic method of solution is that it can also be extended to include both free convection and systems with variable properties, which, as is well known, are usually difficult to analyse theoretically even by approximate techniques.

In an earlier paper on the same subject (Acrivos 1960*a*) the author showed that in the limit of high suction, $B \to -1$, it is indeed possible to derive such an asymptotic expression for general two-dimensional surface geometries and arbitrary but moderate Schmidt numbers and that the rate of mass transfer in the interval $-1 < B < 0$ could well be described by interpolation. It is the purpose of the present article to extend this previous analysis and to investigate the solution of the laminar boundary-layer equations in the limit of high blowing where $B \to \infty$.

2. The solution for large Schmidt numbers

The case $Sc \gg 1$ merits special consideration. This is so, not only because the Schmidt number is indeed very large for most liquid mixtures, but also because it is possible to solve the boundary-layer equations in the limit $Sc \to \infty$ for two-dimensional surfaces with an arbitrary geometry. Thus, as will be shown presently, a closed-form solution which is applicable to many systems of practical interest can readily be obtained for all values of B.

We recall first of all that, when $Sc \gg 1$, the thickness of the diffusion boundary layer is indeed very small and that the resistance to mass transfer in the absence of any interfacial velocity effects is confined to a thin region near the wall where

$$u = (\partial u/\partial y)_{y=0}\, y \equiv \beta(x)\, y, \tag{8}$$

a result which was first proposed by Fage & Falkner (1931) and then proved rigorously by Morgan & Warner (1956). We next observe from equation (5) that, unless B or $(\partial\theta/\partial y)_{y=0}$ are abnormally high, $v \to 0$ as $Sc \to \infty$ at the surface and that the interfacial velocity cannot in general influence the velocity distribution inside the momentum boundary layer, although it can admittedly alter the solution to the convection equation. Following then an established procedure (Acrivos 1960*b*) we begin by stretching the co-ordinates for the region inside the diffusion boundary layer by letting

$$y_1 \equiv Sc^{\frac{1}{3}}y, \quad Sc^{\frac{1}{3}}u = \beta(x)\,y_1, \quad Sc^{\frac{2}{3}}v = -B\left(\frac{\partial\theta}{\partial y_1}\right)_{y_1=0} - \frac{1}{2}\frac{d\beta}{dx}y_1^2, \tag{9}$$

where $\beta(x)$ is kept proportional to the shear stress at the surface in the absence of mass transfer. These new variables are now substituted into equation (3) which becomes

$$\beta y_1\frac{\partial\theta}{\partial x} - \left[B\left(\frac{\partial\theta}{\partial y_1}\right)_{y_1=0} + \frac{1}{2}\frac{d\beta}{dx}y_1^2\right]\frac{\partial\theta}{\partial y_1} = \frac{\partial^2\theta}{\partial y_1^2}, \tag{10}$$

an expression which may be shown rigorously to be the correct asymptotic form of equation (3) as $Sc \to \infty$.

Equation (10) may now be solved with ease. Let

$$\theta = \theta(\eta), \quad \eta = y_1\sqrt{\beta}\bigg/\left[9\int_0^x \sqrt{\beta}\,dz\right]^{\frac{1}{3}}, \tag{11}$$

and
$$b \equiv -\theta'(0) = \left(\frac{jL}{\rho D}\right)\left(\frac{\nu}{U_\infty L}\right)^{\frac{1}{2}}\frac{1}{BSc^{\frac{1}{3}}}\left[9\int_0^x \sqrt{\beta}\,dz\right]^{\frac{1}{3}}\bigg/\sqrt{\beta}, \tag{11a}$$

so that
$$\theta'' + (-Bb + 3\eta^2)\,\theta' = 0, \tag{12}$$

and therefore, as derived by Merk (1959*b*) for the flow past a flat plate,

$$\frac{1}{b} = \int_0^\infty e^{Bbx - x^3}dx, \tag{13}$$

(Mickley, Ross, Squyers & Stewart 1954; Merk 1959*b*; Spalding & Evans 1961), from which b may be determined as a function of B.

This is shown in figure 1, together with the three asymptotic forms of equation (13) which are as follows:

$$\left.\begin{array}{lll} b \to 6^{\frac{1}{3}}(1+B)^{-\frac{1}{3}} & \text{as} & B \to -1, \\ b \to 1{\cdot}120 & \text{as} & B \to 0, \\ b \to 0{\cdot}7425(Bb)^{\frac{1}{4}}\,e^{-0{\cdot}3849(Bb)^{\frac{3}{2}}} & \text{as} & B \to \infty, \end{array}\right\} \tag{14}$$

where b is defined by equation (11*a*). Of particular interest is the fact, clearly shown in figure 1, that, as anticipated in the introduction, the transition from one asymptote to the other is a smooth one, and that if instead of equation (13) only the three asymptotes had been made available it would have been indeed possible merely to draw in, with good accuracy, the curve representing b as a function of B. Another result of considerable practical value is that although the third term of equation (14) is strictly speaking exact only if $B \to \infty$, it is, as can be seen from figure 1, of acceptable accuracy even when B is as low as 2, which, as will be shown later on in this paper, is fortunately not an isolated property of the solution for large Schmidt numbers, but a more general characteristic of such fluid-mechanical systems.

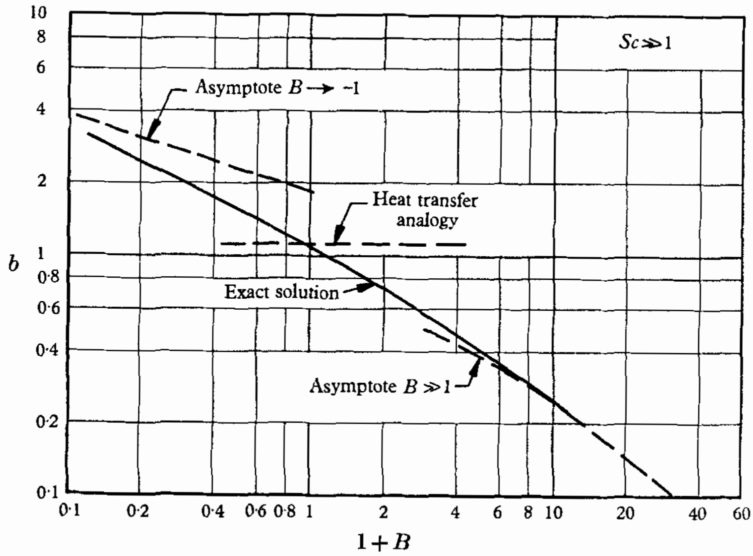

FIGURE 1. Forced convection for large Sc; b is defined by equation (11a).

3. The solution for $B \rightarrow \infty$ and for moderate or small Schmidt numbers

The previous section has dealt with the high Schmidt number solution and has been restricted to cases where the interfacial velocity was always so small that its effect on the velocity distribution in the momentum boundary layer could be neglected. This is so only if the Schmidt number is sufficiently large that

$$BbSc^{-\frac{2}{3}} \ll 1 \tag{15}$$

for all B of interest.† It is clear, however, both from the above and from equation (5), that if Sc is moderate or small then the interfacial velocity can indeed be large and that its influence on the velocity distribution cannot be overlooked.

This problem, which is of considerable practical interest since Sc is around unity or smaller for most gaseous mixtures, was analysed in an earlier paper (Acrivos 1960a), where it was shown that, for arbitrary $U(x)$, the laminar boundary-layer equations could be solved in a closed form for $B \rightarrow -1$ and for arbitrary but moderate Sc.‡ It is the purpose of the present section to complete this theoretical study and to consider the asymptotic solution as $B \rightarrow \infty$.

First of all, the boundary-layer equations are rearranged by using the transformations proposed by Meksyn (1948), by Görtler (1957) and by Merk (1959a). Thus we write

$$\xi \equiv \int_0^x U(x)\,dx, \quad \eta \equiv Uy(2\xi)^{-\frac{1}{2}}, \quad \Lambda \equiv 2\xi d(\ln U)/d\xi, \tag{16}$$

† It should be kept in mind that: $\lim_{B\to\infty} [\lim_{Sc\to\infty} j] \neq \lim_{Sc\to\infty} [\lim_{B\to\infty} j]$.

‡ The last restriction is necessary because: $\lim_{Sc\to\infty} [\lim_{B\to -1} j] \neq \lim_{B\to -1} [\lim_{Sc\to\infty} j]$.

and expand u, v and θ as follows:

$$\left.\begin{aligned} &u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \quad \psi(x,y) = (2\xi)^{\frac{1}{2}} f(\xi,\eta), \\ \text{where} \quad &f(\xi,\eta) = f_0(\Lambda,\eta) + 2\xi(d\Lambda/d\xi) f_1(\Lambda,\eta) + \ldots, \\ \text{and} \quad &\theta(\xi,\eta) = \theta_0(\Lambda,\eta) + 2\xi(d\Lambda/d\xi)\,\theta_1(\Lambda,\eta) + \ldots. \end{aligned}\right\} \tag{17}$$

These, when substituted into the original boundary-layer equations, simplify to

$$f_0''' + f_0 f_0'' + \Lambda(1 - f_0'^2) = 0, \tag{18a}$$

$$\theta_0'' + Sc f_0 \theta_0' = 0, \tag{18b}$$

with the following boundary conditions:

$$\left.\begin{aligned} &\text{at } \eta = 0\text{:} \quad \theta_0 = 1, \quad f_0' = 0, \quad f_0 \equiv -\alpha = (B/Sc)\,\theta_0'; \\ &\text{at } \eta = \infty\text{:} \quad \theta_0 = 0, \quad f_0' = 1. \end{aligned}\right\} \tag{19}$$

It should be noted of course that, as shown by Merk (1959*a*), additional expressions must be written down for the remaining functions f_1, θ_1, etc., but since the analysis quickly becomes far too complicated, we shall content ourselves with the solution of equations (18) and (19). Strictly speaking, therefore, the present development is not as general as the earlier one for $B \to -1$ (Acrivos 1960*a*), since it is exact only for wedge-like surfaces for which Λ is constant, proportional to the wedge angle. Fortunately, however, this limitation is not at all as serious as it might appear at first glance because it has been established (Merk 1959*a*) that the wedge-type simplification to the boundary-layer equations as used above, does lead to reasonably reliable results for the rate of pure heat transfer, except near the separation point where, on account of the fact that the location of separation cannot be predicted with any great accuracy by this approximate procedure, the approximation is rather poor. It is felt, though, that in the case of mass transfer with large blowing the overall agreement between the results of the wedge-type method and those of the exact solution should be improved somewhat, especially since the location of separation, which as $B \to \infty$ moves closer and closer to the point of minimum pressure, can now be determined with certainty.

Let us begin then with equations (18*a*) and (19), keeping in mind that we are seeking their solution as $\alpha \to \infty$. This appears easy at first, since the transformations

$$z \equiv \eta/\alpha \quad \text{and} \quad f_0 = -\alpha + \alpha\phi(z) \tag{20}$$

reduce equations (18*a*) and (19) to

$$\left.\begin{aligned} &\alpha^{-2}\phi''' - (1-\phi)\,\phi'' + \Lambda(1-\phi'^2) = 0, \\ &\phi(0) = \phi'(0) = 0, \quad \phi'(\infty) = 1, \end{aligned}\right\} \tag{21}$$

which seems to be ideally suited for a perturbation solution. Unfortunately, however, the perturbation is of a singular type, since no matter what the value of α, one would expect the term $\phi'''\alpha^{-2}$ to be larger than $(1-\phi)\,\phi''$ if ϕ were close enough to unity. It follows therefore that the equation will have to be attacked by a singular perturbation technique, analogous to the one employed by Kaplun

& Lagerstrom (1957) and by Proudman & Pearson (1957) in their successful solution of certain classical fluid-mechanical problems.

It is for this purpose that the variables are once again transformed by letting

$$\lambda \equiv 1-\phi \quad \text{and} \quad \omega \equiv 1-(d\lambda/dz)^2, \quad \text{where} \quad d\lambda/dz \leqslant 0, \tag{22}$$

so that equation (21) is changed into

$$\frac{(1-\omega)^{\frac{1}{2}}}{\alpha^2}\frac{d^2\omega}{d\lambda^2}+\lambda\frac{d\omega}{d\lambda}-2\Lambda\omega = 0, \tag{23}$$

with the boundary conditions $\omega(1) = 1$, $\omega(-\infty) = 0$. Proceeding now according to a well established rule (Proudman & Pearson 1957), we first construct an *inner* perturbation expansion for ω which will satisfy the boundary condition $\omega(1) = 1$, but not necessarily the requirement $\omega(-\infty) = 0$. This can be accomplished in a straightforward manner by letting

$$\omega = \omega_0+\alpha^{-2}\omega_1+\ldots, \tag{24}$$

and then substituting into equation (23). It is found that

$$\lambda(d\omega_0/d\lambda)-2\Lambda\omega_0 = 0, \quad \lambda(d\omega_1/d\lambda)-2\Lambda\omega_1 = -(1-\omega_0)^{\frac{1}{2}}(d^2\omega_0/d\lambda^2), \quad \text{etc.,}$$

from which it readily follows that

$$\omega = \lambda^{2\Lambda}+\frac{1}{\alpha^2}2\Lambda(2\Lambda-1)\lambda^{2\Lambda}\int_\lambda^1\frac{(1-x^{2\Lambda})^{\frac{1}{2}}}{x^3}dx+O(1/\alpha^4). \tag{25}$$

It is immediately obvious at this point that equation (25) does not satisfy the boundary condition $\omega(-\infty) = 0$. Nevertheless, the function

$$\omega = \lambda^{2\Lambda} \quad \text{for} \quad 0 \leqslant \lambda \leqslant 1, \qquad \omega = 0 \quad \text{for} \quad \lambda < 0$$

would indeed be a perfectly acceptable solution to equation (23) if α were sufficiently large, except for the discontinuities in the derivatives when $\lambda \sim 0$. This would mean then that whereas it is permissible to neglect the viscous term in equation (23) throughout most of the boundary layer if $\alpha \gg 1$, there is clearly an important shear layer of dimensions α^{-1} in the λ-co-ordinate which is located, not at the surface as is usually the case with such systems, but at an appreciable distance from it where $\omega \sim 0$. We conclude therefore that the appropriate *outer* expansion for the function—denoted here as $\Omega(t)$—which would satisfy the boundary condition $\Omega(-\infty) = 0$, would be of the form

$$\Omega(t) = C_1(\alpha)\,\Omega_1(t)+C_2(\alpha)\,\Omega_2(t)+\ldots, \tag{26}$$

where $\quad t \equiv -\alpha\lambda, \quad \lim_{\alpha\to\infty} C_1(\alpha) = 0, \quad \lim_{\alpha\to\infty}\frac{C_2(\alpha)}{C_1(\alpha)} = 0, \quad$ etc.

It is clear then, from equations (23) and (26), that if α is large enough $\Omega_1(t)$ must satisfy the equation

$$\frac{d^2\Omega_1}{dt^2}+t\frac{d\Omega_1}{dt}-2\Lambda\Omega_1 = 0, \tag{27}$$

which fortunately has the simple closed form solution (Erdélyi *et al.* 1953, vol. II, p. 116):

$$\Omega_1 = \int_t^\infty e^{-\tau^2/2}(\tau-t)^{2\Lambda}\,d\tau. \tag{28}$$

This must finally be matched to the inner expression as given by equation (25) in such a way that $\lim_{t\to-\infty} \Omega = \lim_{\lambda\to 0} \omega$. For $\alpha \gg 1$, however,

$$\Omega \to C_1(\alpha)\{(2\pi)^{\frac{1}{2}}(\alpha\lambda)^{2\Lambda} + (2\pi)^{\frac{1}{2}}\Lambda(2\Lambda-1)(\alpha\lambda)^{2\Lambda-2}+\ldots\} \quad \text{as} \quad t \equiv -\alpha\lambda \to -\infty,$$

while
$$\omega \to \lambda^{2\Lambda} + \Lambda(2\Lambda-1)\lambda^{2\Lambda-2}/\alpha^2 + \ldots \quad \text{as} \quad \lambda \to 0$$

so that the matching is exact for large α if

$$C_1(\alpha) = (2\pi)^{-\frac{1}{2}}\alpha^{-2\Lambda}. \tag{29}$$

We now turn our attention to the convection equation, equation (18*b*), which, in the new variables λ and z, becomes

$$\frac{d^2\theta_0}{dz^2} - \alpha^2\lambda Sc\frac{d\theta_0}{dz} = 0, \tag{30}$$

whence
$$-\left(\frac{dz}{d\theta_0}\right)_{z=0} = \int_0^\infty \exp\left(Sc\alpha^2\int_0^z \lambda\,dz\right)dz. \tag{31}$$

This may be evaluated in the following manner. First of all,

$$\int_0^z \lambda\,dz = \int_0^{z^*}\lambda\,dz + \frac{1}{2}\left(\frac{d\lambda}{dz}\right)_{z=z^*}(z-z^*)^2 + \frac{1}{3!}\left(\frac{d^2\lambda}{dz^2}\right)_{z=z^*}(z-z^*)^3+\ldots, \tag{32}$$

where, by definition,

$$\lambda(z^*) = 0. \tag{33}$$

Because of equations (22), (28) and (29), however,

$$\frac{d\lambda}{dz} \equiv -(1-\Omega)^{\frac{1}{2}}, \quad \left(\frac{d\lambda}{dz}\right)_{z=z^*} = -1 + O(\alpha^{-2\Lambda})$$

and in general
$$\left(\frac{d^n\lambda}{dz^n}\right)_{z=z^*} = O(\alpha^{-2\Lambda-1+n}) \quad \text{for} \quad n \geqslant 1.$$

Therefore, with the substitution $\eta \equiv z\alpha$, equation (31) rearranges into

$$-\left(\frac{d\eta}{d\theta_0}\right)_{\eta=0}\exp\left\{-Sc\alpha^2\int_0^{z^*}\lambda\,dz\right\}$$
$$= \int_0^\infty d\eta[\exp\{-\tfrac{1}{2}Sc(\eta-\eta^*)^2\} + O(\alpha^{-2\Lambda})] \to (2\pi/Sc)^{\frac{1}{2}} + O(\alpha^{-2\Lambda}), \tag{34}$$

since $\eta^* \equiv az^* \gg 1$ if $\alpha \gg 1$.

It remains then to determine the exponential term on the left-hand side of equation (34). First of all,

$$\int_0^{z^*}\lambda\,dz = \int_0^1 \frac{\lambda\,d\lambda}{(1-\omega)^{\frac{1}{2}}} \equiv I,$$

where, for $\lambda > O(1/\alpha)$, ω is given by equation (25), whereas for $\lambda < O(1/\alpha)$ equation (28) must be employed. This means that

$$I = \int_{1/\alpha}^1 \frac{\lambda\,d\lambda}{(1-\omega_0-\alpha^{-2}\omega_1)^{\frac{1}{2}}} + \int_0^{1/\alpha}\frac{\lambda\,d\lambda}{(1-\Omega)^{\frac{1}{2}}},$$

which can be simplified still further since

$$\int_{1/\alpha}^{1} \frac{\lambda\, d\lambda}{(1-\omega_0-\alpha^{-2}\omega_1)^{\frac{1}{2}}} = \int_{1/\alpha}^{1} \frac{\lambda\, d\lambda}{(1-\omega_0)^{\frac{1}{2}}} + \frac{1}{2\alpha^2}\int_{1/\alpha}^{1} \frac{\lambda\omega_1\, d\lambda}{(1-\omega_0)^{\frac{3}{2}}} + O(\alpha^{-4})$$

and
$$\int_{0}^{1/\alpha} \frac{\lambda\, d\lambda}{(1-\Omega)^{\frac{1}{2}}} = \int_{0}^{1/\alpha} \frac{\lambda\, d\lambda}{(1-\omega_0)^{\frac{1}{2}}} + \frac{1}{2}\int_{0}^{1/\alpha} \frac{\lambda(\Omega-\omega_0)}{(1-\omega_0)^{\frac{3}{2}}}\, d\lambda + O(<\alpha^{-2\Lambda-2}).$$

Therefore,

$$I = \int_{0}^{1} \frac{\lambda\, d\lambda}{(1-\omega_0)^{\frac{1}{2}}} + \frac{1}{2\alpha^2}\int_{0}^{1} \frac{\lambda\omega_1\, d\lambda}{(1-\omega_0)^{\frac{3}{2}}} - \frac{1}{2\alpha^2}\int_{0}^{1/\alpha} \frac{\lambda\omega_1\, d\lambda}{(1-\omega_0)^{\frac{3}{2}}}$$
$$+\frac{1}{2}\int_{0}^{1/\alpha} \frac{\lambda(\Omega-\omega_0)}{(1-\omega_0)^{\frac{3}{2}}}\, d\lambda + O(\alpha^{-4}) + O(<\alpha^{-2\Lambda-2}).$$

Finally it can be shown that the third and fourth terms of the above expression are both of order $\alpha^{-2\Lambda-2}$ so that

$$I = \int_{0}^{1} \frac{\lambda\, d\lambda}{(1-\omega_0)^{\frac{1}{2}}} + \frac{1}{2\alpha^2}\int_{0}^{1} \frac{\lambda\omega_1\, d\lambda}{(1-\omega_0)^{\frac{3}{2}}} + O(\alpha^{-4}) + O(\alpha^{-2\Lambda-2}), \tag{35}$$

where, from equation (25),

$$\omega_0 = \lambda^{2\Lambda} \quad \text{and} \quad \omega_1 = 2\Lambda(2\Lambda-1)\,\lambda^{2\Lambda}\int_{\lambda}^{1} \frac{(1-x^{2\Lambda})^{\frac{1}{2}}}{x^3}\, dx.$$

The first term on the right-hand side of equation (35) can of course be easily integrated, while, as explained in the appendix, the second term can also be expressed in terms of tabulated functions. The result is

$$\int_{0}^{1} \frac{\lambda\, d\lambda}{(1-\omega_0)^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(1+\Lambda^{-1})}{\Gamma(\frac{1}{2}+\Lambda^{-1})} \quad \text{and} \quad \int_{0}^{1} \frac{\lambda\omega_1\, d\lambda}{(1-\omega_0)^{\frac{3}{2}}} = \frac{(2\Lambda-1)}{(2-\Lambda)}[\Psi(1+\Lambda^{-1})-\Psi(\tfrac{3}{2})],$$

where Ψ is the logarithmic derivative of the gamma function. One can conclude then from equation (34) and the above that

$$-\left(\frac{d\theta_0}{d\eta}\right)_{\eta=0} = \left(\frac{Sc}{2\pi}\right)^{\frac{1}{2}} \exp\left[-Sc\left\{\alpha^2\frac{\sqrt{\pi}}{2}\frac{\Gamma(1+\Lambda^{-1})}{\Gamma(\frac{1}{2}+\Lambda^{-1})}\right.\right.$$
$$\left.\left.+\frac{(\Lambda-\frac{1}{2})}{(2-\Lambda)}[\Psi(1+\Lambda^{-1})-0{\cdot}03648]\right\}\right] + O(\alpha^{-k}), \tag{36}$$

with
$$k \equiv \min(2\Lambda, 2), \tag{37}$$

while, by combining equations (7), (16) and (36), one can derive the asymptotic expression for the rate of mass transfer:

$$j_1 \equiv \frac{jL}{\rho D}\left(\frac{\nu}{U_\infty L}\right)^{\frac{1}{2}}\left\{\left(2\int_0^x U\, dx\right)^{\frac{1}{2}}\Big/U(x)\right\}\left(\frac{\sqrt{\pi}\,\Gamma(1+\Lambda^{-1})}{2Sc\Gamma(\frac{1}{2}+\Lambda^{-1})}\right)^{\frac{1}{2}} = \alpha_1(B_1) + O(\alpha_1^{-k}), \tag{38}$$

where α_1, defined as

$$\alpha_1 \equiv \alpha\left(\frac{\sqrt{\pi}\,\Gamma(1+\Lambda^{-1})}{2\Gamma(\frac{1}{2}+\Lambda^{-1})}Sc\right)^{\frac{1}{2}}, \tag{38a}$$

is given implicitly, with

$$\alpha_1 = B_1 \exp(-\alpha_1^2) \tag{39}$$

by $$B_1 \equiv B\left(\frac{\Gamma(1+\Lambda^{-1})}{4\sqrt{\pi}\,\Gamma(\frac{1}{2}+\Lambda^{-1})}\right)^{\frac{1}{2}} \exp\left\{-Sc\frac{(\Lambda-\frac{1}{2})}{(2-\Lambda)}[\Psi(1+\Lambda^{-1})-0{\cdot}03648]\right\}. \tag{40}$$

Equation (38) then provides us with the desired asymptotic result in a surprisingly simple form. Of particular interest is the fact that, had the viscous part of equation (21) been completely neglected, equation (38) would still have held except that the exponential factor in equation (40) would have been missing. As this term is rather close to unity in the general case, one can infer that the viscous effects will become progressively less important at high blowing which is of course to be expected, since, with increased blowing, the shear layer is gradually displaced from the immediate neighbourhood of the surface.

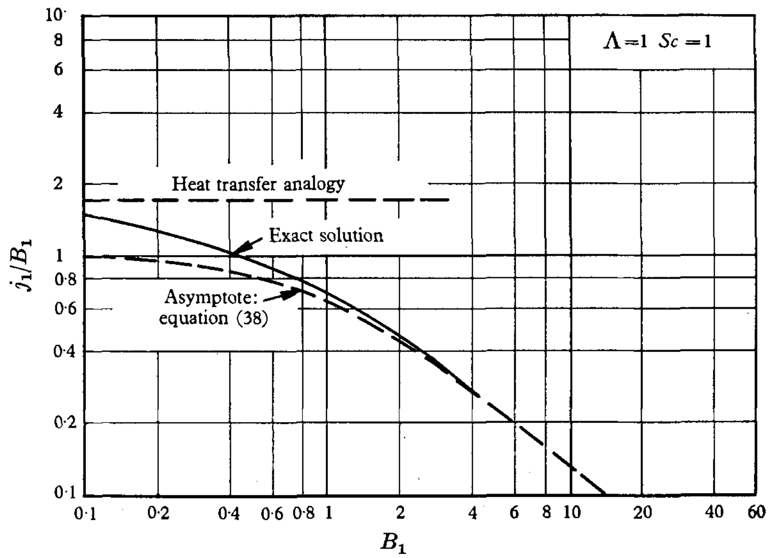

FIGURE 2. Stagnation-flow forced convection; j_1 and B_1 are defined respectively by equations (38) and (40).

It remains now to compare equation (38) with some exact numerical values (Livingood & Donoughe 1955; Eckert & Hartnett 1957; Stewart & Prober 1961). Typical results are those for the stagnation flow ($\Lambda = 1$) with $Sc = 1$ and are shown in figure 2 together with the asymptote derived above. It is immediately apparent once again that, as with the large Schmidt number solution reported in § 2 and presented in figure 1, the transition from the pure heat-transfer result into equation (38) is a smooth one, that the asymptotic expression for the rate of mass transfer describes fairly well the exact function even for relatively low values of B and that, for this example at any rate, equation (38) is reasonably accurate for $B_1 > 1$ or $B > 3$. A similar conclusion can also be reached with respect to the system $\Lambda = \frac{2}{3}$, $Sc = 0{\cdot}7$, which is shown in figure 3.

We close this section by presenting briefly the solution to equations (18) and (19) for $Sc \to 0$. It is realized of course that, contrary to pure heat transfer, the case $Sc \to 0$ is not of too much practical interest because the Schmidt number appears to be always higher than about 0·2 for physical systems. Nevertheless, since, as we shall show below, equations (18) and (19) can indeed be solved in closed form for arbitrary B if $Sc \to 0$, the solution thus derived can be of considerable theoretical value in that it can provide us with a quantitative description of the mass-transfer rate function j_1 in the interval between the pure heat-transfer analogy and the $B \to \infty$ asymptote.

The chief characteristic of the $Sc \to 0$ solution to equations (18) and (19) is that, although $\alpha \to \infty$ for all $B > 0$, $\alpha\sqrt{Sc}$ remains, as will be presently demon-

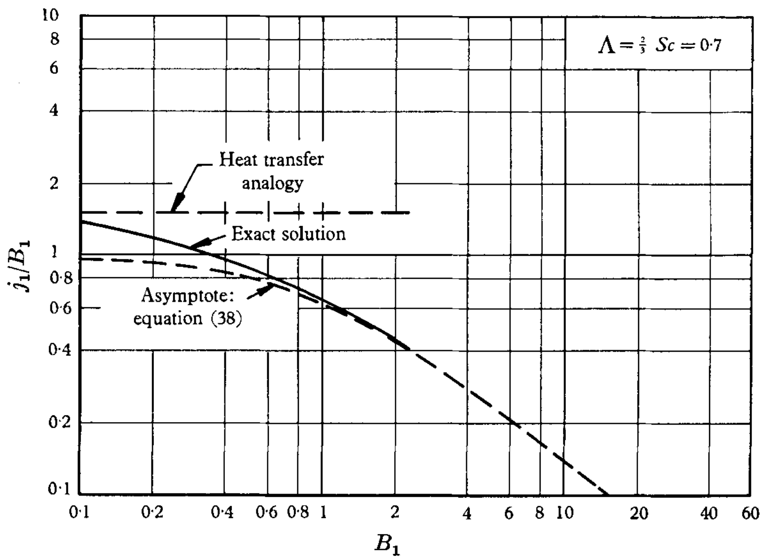

FIGURE 3. Forced convection from a wedge at angle $\frac{2}{3}\pi$; j_1 and B_1 are defined respectively by equations (38) and (40).

strated, a unique function of Λ and B. Let us for the moment then assume that $\alpha \to \infty$ for $Sc \to 0$. It follows from equations (25) and (31) that

$$-\left(\frac{dz}{d\theta_0}\right)_{z=0} = \int_0^\infty \exp\left\{\alpha^2 Sc \int_0^z \lambda\, dx\right\} dz,$$

where λ is given exactly by

$$\frac{d\lambda}{dz} = -(1-\lambda^{2\Lambda})^{\frac{1}{2}} \quad \text{for} \quad 0 < \lambda < 1, \qquad \frac{d\lambda}{dz} = -1 \quad \text{for} \quad \lambda < 0.$$

Therefore

$$-\left(\frac{dz}{d\theta_0}\right)_{z=0} = \exp\left(\alpha^2 Sc \int_0^1 \frac{\lambda\, d\lambda}{(1-\lambda^{2\Lambda})^{\frac{1}{2}}}\right)$$
$$\times \left[\left(\frac{\pi}{2\alpha^2 Sc}\right)^{\frac{1}{2}} + \int_0^1 \frac{d\lambda}{(1-\lambda^{2\Lambda})^{\frac{1}{2}}} \exp\left(-\alpha^2 Sc \int_0^\lambda \frac{x\, dx}{(1-x^{2\Lambda})^{\frac{1}{2}}}\right)\right] \equiv \frac{B}{\alpha^2 Sc}.$$

Consequently, if α_1 is defined as in equation (38*a*) and B_1 as in equation (40) with the exponential term missing, the more general form of equation (39), valid for all α_1 but restricted to $Sc \ll 1$, is

$$B_1 = \alpha_1^2 \exp(\alpha_1^2)\left[\frac{1}{2\alpha_1}+\frac{1}{A(2\pi)^{\frac{1}{2}}}\int_0^1 \frac{d\lambda}{(1-\lambda^{2\Lambda})^{\frac{1}{2}}}\exp\left\{-\frac{\alpha_1^2}{A^2}\int_0^\lambda \frac{x\,dx}{(1-x^{2\Lambda})^{\frac{1}{2}}}\right\}\right], \quad (39a)$$

where

$$A \equiv \left(\frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma(1+\Lambda^{-1})}{\Gamma(\frac{1}{2}+\Lambda^{-1})}\right)^{\frac{1}{2}}.$$

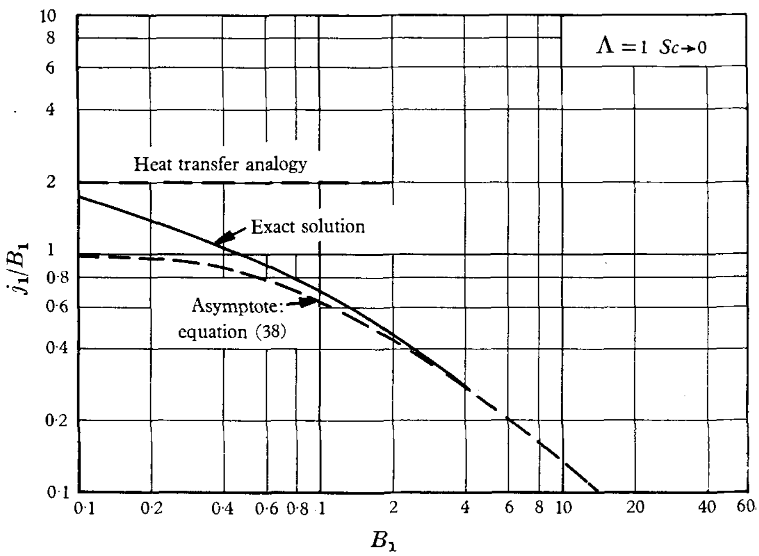

FIGURE 4. Stagnation-flow forced convection for low Sc; j_1 and B_1 are defined respectively by equations (38) and (40).

Equation (39*a*) reduces of course to equation (39) as $\alpha_1 \to \infty$, while, for $\alpha_1 \to 0$, it simplifies to

$$B_1 = \alpha_1^2 \exp(\alpha_1^2)\left[\frac{1}{2\alpha_1}+0{\cdot}7513\frac{\Gamma(1+\frac{1}{2}\Lambda^{-1})}{\Gamma(\frac{1}{2}+\frac{1}{2}\Lambda^{-1})}\left(\frac{\Gamma(\frac{1}{2}+\Lambda^{-1})}{\Gamma(1+\Lambda^{-1})}\right)^{\frac{1}{2}}+O(\alpha_1^2)\right].$$

The function $\alpha_1(B_1)$ for $\Lambda = 1$ is shown plotted in figure 4. It is seen that, as was the case with the previous examples, the transition from one asymptote to the other is smooth and that equation (39) becomes accurate when $B_1 > 1$, a result which, naturally, further strengthens our argument about the usefulness of our asymptotic solution.

4. The solution for $\Lambda = 0$

Since the error in the asymptotic expression given by equation (38) is $O(\alpha^{-2\Lambda})$ if $\Lambda \leqslant 1$, it is obvious that the mathematical development presented in the previous section will be applicable only so long as the pressure gradient remains favourable and $\Lambda > 0$. It should be expected on the other hand that, as $\Lambda \to 0$, equation (38) would become less and less accurate for moderate B, and that for

$\Lambda = 0$ a completely different solution would then hold. This important special case, which was first studied in considerable detail by Mickley *et al.* (1954), is discussed briefly below.

When $\Lambda = 0$, equation (18a) reduces to

$$\left.\begin{aligned} f''' + ff'' &= 0, \\ \text{with} \qquad f(0) = -\alpha, \quad f'(0) &= 0, \quad f'(\infty) = 1, \end{aligned}\right\} \tag{41}$$

which has already been solved numerically (Emmons & Leigh 1953). An interesting and rather unique property of this solution is that the boundary layer becomes detached from the surface when $\alpha = 0{\cdot}5758$, which means that, because of equations (7), (16) and (19),

$$\frac{\sqrt{2jL}}{\rho DB} \to \frac{0{\cdot}8758}{B} Sc \quad \text{as} \quad B \to \infty. \tag{42}$$

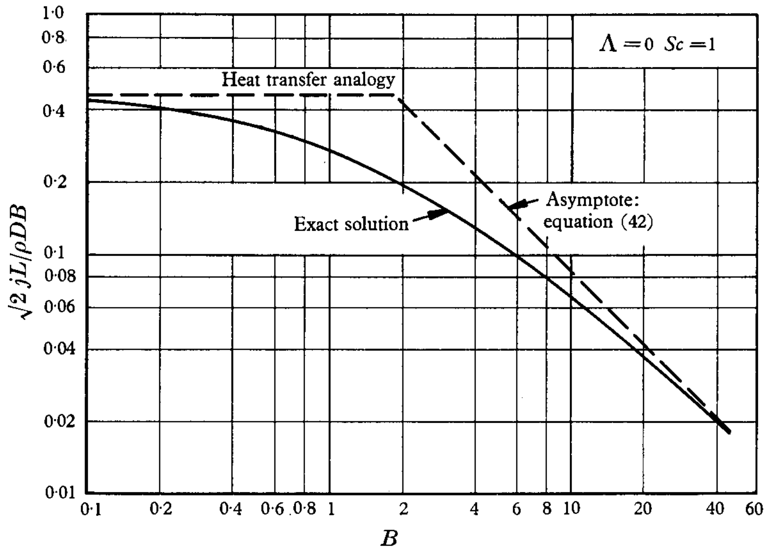

FIGURE 5. Forced convection from a flat plate.

The behaviour of the function $\sqrt{2jL}/\rho DB$ for $Sc = 1$, as obtained from the Emmons & Leigh (1953) numerical calculations, is presented in figure 5 together with the appropriate asymptotes. Again, the transition from the pure heat-transfer result into the large blowing asymptotic expression is smooth, although equation (42) becomes in this case accurate only if $B > 10$, which is somewhat larger than the corresponding value for B in the earlier examples.

5. Mixtures with variable properties

The assumption of constant fluid properties, which has been used up to now in our development, is in general correct only so long as $B \sim 0$, for under such conditions the composition of the fluid obviously remains relatively constant throughout the boundary layer. In the limit $B \to \infty$, however, this may not necessarily be the case. Thus, in gases for example, the mass density is for practical purposes directly proportional to the molecular weight, which means that, if the components of the mixture are widely different in molecular size, the variation of ρ, as well as of μ and D, across the boundary layer may indeed be appreciable and cause a considerable error in the predictions of the constant-property model. We shall therefore consider here briefly the variable-property system and outline a method by which the results arrived at previously may be extended.

Let us then begin with the appropriate two-dimensional boundary-layer equations (Lees 1956; Acrivos 1960*a*) analogous to equations (1) to (3):

$$\left.\begin{aligned}&\frac{\partial}{\partial x}(\bar{\rho}u)+\frac{\partial}{\partial y}(\bar{\rho}v)=0,\\&\bar{\rho}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\bar{\rho}_\infty U\frac{dU}{dx}+\frac{\partial}{\partial y}\left(\bar{\mu}\frac{\partial u}{\partial y}\right),\\&\bar{\rho}\left(u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}\right)=\frac{1}{Sc}\frac{\partial}{\partial y}\left(\bar{\rho}\bar{D}\frac{\partial \theta}{\partial y}\right),\end{aligned}\right\}\tag{43}$$

in which the dimensionless variables v and y are defined in terms of the fluid properties *at the surface*, and where

$$\bar{\rho}\equiv\rho/\rho_s,\quad \bar{\rho}_\infty=\rho_\infty/\rho_s,\quad \bar{D}\equiv D/D_s,\quad \bar{\mu}=\mu/\mu_s.$$

Our analysis will be limited to systems with moderate or low Schmidt numbers since the case $Sc \gg 1$ may be handled, without difficulty, by a straightforward extension of an existing procedure for solving a somewhat related heat transfer problem (Acrivos 1960*b*). ρ_∞ is taken as constant.

Equations (43) are once again rearranged by the introduction of the new variables, similar to those defined by equations (16) and (17), to give

$$\left.\begin{aligned}&\xi\equiv\int_0^x U\,dx,\quad \eta\equiv\frac{Uy\bar{\rho}_\infty^{\frac{1}{2}}}{(2\xi)^{\frac{1}{2}}},\quad \Lambda\equiv 2\xi\frac{d\ln U}{d\xi},\\&u=\bar{\rho}_\infty^{\frac{1}{2}}U\left(\frac{\partial f}{\partial \eta}\right)_\xi,\quad H\equiv\int_0^\theta\bar{\rho}\bar{D}d\theta\Big/\int_0^1\bar{\rho}\bar{D}d\theta.\end{aligned}\right\}\tag{44}$$

It can now be shown that if, as was explained earlier in connexion with equation (18), the functions f and H are assumed to be independent of ξ as a first approximation, equations (43) and (44) may be combined to give

$$\left.\begin{aligned}&\bar{\rho}_\infty^{\frac{1}{2}}\frac{d}{d\eta}(\bar{\mu}f'')+f''\left[-\alpha+\int_0^\eta\bar{\rho}f'\,d\eta\right]+\Lambda\{1-\bar{\rho}f'^2\}=0,\\&\bar{\rho}_\infty^{\frac{1}{2}}\frac{d^2H}{d\eta^2}+\frac{Sc}{\bar{\rho}\bar{D}}\frac{dH}{d\eta}\left[-\alpha+\int_0^\eta\bar{\rho}f'\,d\eta\right]=0,\end{aligned}\right\}\tag{45}$$

where $\bar{\rho}$, $\bar{D}$ and $\bar{\mu}$ are in general known functions of H. The boundary conditions are:

$$\left.\begin{aligned}&\text{at }\eta=0:\quad H=1,\quad f'=0,\quad f=-\alpha=\frac{B}{Sc}\bar{\rho}_\infty^{\frac{1}{2}}H'\int_0^1\bar{\rho}\bar{D}d\theta;\\&\text{at }\eta=\infty:\quad H=0,\quad f'=\bar{\rho}_\infty^{-\frac{1}{2}}.\end{aligned}\right\}\tag{46}$$

As before, the solution to equations (45) in the limit $\alpha \to \infty$ will be obtained by a singular perturbation expansion. We note first, however, that in the so-called 'inner' region where both the molecular diffusion and shear effects are negligible, $H \equiv 1$, and therefore $\bar{\rho} = \bar{\mu} = 1$. One can conclude then that the 'inner' perturbation expansion of equation (45) is essentially the same as that of equation (18*a*), which means that, in view of equation (25),

$$1 - f'^2 = \lambda^{2\Lambda} + \frac{\bar{\rho}_\infty^{\frac{1}{2}}}{\alpha^2} 2\Lambda(2\Lambda - 1)\lambda^{2\Lambda} \int_\lambda^1 \frac{(1 - x^{2\Lambda})^{\frac{1}{2}}}{x^3} dx + O\left(\frac{1}{\alpha^4}\right), \tag{47}$$

where

$$\lambda \equiv 1 - \frac{1}{\alpha} \int_0^\eta \bar{\rho} f' \, d\eta.$$

The function given by equation (47) must now be matched, as explained earlier, to an 'outer' perturbation solution which is obtained as follows. Let

$$\eta_1 \equiv \eta - \eta^*,$$

where, by definition,

$$\int_0^{\eta^*} \bar{\rho} f' \, d\eta = \alpha. \tag{48}$$

This means that, in the new variable η_1, equations (45) become

$$\bar{\rho}_\infty^{\frac{1}{2}} \frac{d}{d\eta_1} (\bar{\mu} f'') + f'' \int_0^{\eta_1} \bar{\rho} f' \, d\eta_1 + \Lambda\{1 - \bar{\rho} f'^2\} = 0 \tag{49}$$

and

$$\frac{d^2 H}{d\eta_1^2} + \frac{Sc}{\bar{\rho}_\infty^{\frac{1}{2}} \bar{\rho} \bar{D}} \frac{dH}{d\eta_1} \int_0^{\eta_1} \bar{\rho} f' \, d\eta_1 = 0, \tag{49a}$$

where the boundary conditions are to a first approximation as $\alpha \to \infty$:

$$\left.\begin{aligned} &\text{at } \eta_1 = -\infty, \quad f' = 1, \quad H = 1; \qquad \text{at } \eta_1 = 0, \quad f = 0; \\ &\text{at } \eta_1 = \infty, \quad f' = \bar{\rho}_\infty^{-\frac{1}{2}}, \quad H = 0; \end{aligned}\right\} \tag{50}$$

on account of the required matching between the solution to equation (49) and the function given by equation (47). Clearly now, equations (49) and (50), which do not contain either α or B as parameters, must in the general case be solved numerically since an analytic solution does not exist. We shall show, however, that the general form of equation (38) for the rate of mass transfer is still retained even though the fluid properties may be arbitrary functions of composition.

We note first of all from equation (49*a*) that

$$\left(\frac{dH}{d\eta}\right)_{\eta=0} = \left(\frac{dH}{d\eta_1}\right)_{\eta_1=0} \exp\left\{-\frac{Sc}{\bar{\rho}_\infty^{\frac{1}{2}}} \int_0^{-\eta^*} \frac{1}{\bar{\rho}\bar{D}} \left(\int_0^x \bar{\rho} f' \, dx_1\right) dx\right\}. \tag{51}$$

On the other hand, $\eta^* \to \infty$ as $\alpha \to \infty$ and, in view of the results previously obtained,

$$\int_0^{-\eta^*} \frac{1}{\bar{\rho}\bar{D}} \left(\int_0^x \bar{\rho} f' \, dx_1\right) dx \to \tfrac{1}{2}\alpha^2 \sqrt{\pi} \frac{\Gamma(1 + \Lambda^{-1})}{\Gamma(\frac{1}{2} + \Lambda^{-1})} + \bar{\rho}_\infty^{\frac{1}{2}} \frac{(\Lambda - \frac{1}{2})}{(2 - \Lambda)} [\Psi(1 + \Lambda^{-1}) - 0{\cdot}03648] + E,$$

with

$$E \equiv \int_{-\infty}^0 \frac{1 - \bar{\rho}^2 \bar{D} f'}{\bar{\rho}\bar{D}} \left(\int_{\eta_1}^0 \bar{\rho} f' \, dx\right) d\eta_1, \tag{52}$$

where, it should be noted, E will have to be computed from the solution of equations (49) and (50). It follows, therefore, that equation (38) can be generalized for a variable-proper into

$$\jmath_1 \equiv \frac{jL}{\rho_s D_s}\left(\frac{\nu_s}{U_\infty L}\right)^{\frac{1}{2}} \left\{\frac{\left(2\int_0^x U\,dx\right)^{\frac{1}{2}}}{\bar{\rho}_\infty^{\frac{1}{2}}\, U(x)}\right\} \left(\frac{\sqrt{\pi}\,\Gamma(1+\Lambda^{-1})}{2Sc\,\Gamma(\frac{1}{2}+\Lambda^{-1})}\right)^{\frac{1}{2}} = \alpha_1(B_1) + O(\alpha^{-k}), \quad (53)$$

where, as before, $k = \min(2\Lambda, 2)$ while α_1 is given implicitly in terms of B_1 by

$$\alpha_1 = B_1 \exp(-\alpha_1^2)$$

with

$$B_1 \equiv -\left(\frac{dH}{d\eta_1}\right)_{\eta_1=0}\left(\frac{2\pi}{Sc}\right)^{\frac{1}{2}} B\left(\frac{\Gamma(1+\Lambda^{-1})}{4\sqrt{\pi}\,\Gamma(\frac{1}{2}+\Lambda^{-1})}\right)^{\frac{1}{2}}\left(\int_0^1 \bar{\rho}\bar{D}\,d\theta\right)\bar{\rho}_\infty^{\frac{1}{2}}$$
$$\times \exp\left\{-Sc\frac{(\Lambda-\frac{1}{2})}{(2-\Lambda)}[\Psi(1+\Lambda^{-1})-0{\cdot}03648]-\frac{ScE}{\bar{\rho}_\infty^{\frac{1}{2}}}\right\}. \quad (54)$$

We can clearly see then that, although no analytic solution can be obtained in the general case since the definition of B_1 involves two parameters, $H'(0)$ and E, which must be evaluated numerically from equations (49) to (51), the asymptotic expression for the rate of mass transfer with $B \gg 1$ retains exactly the same form as equation (38) even when the fluid parameters are arbitrary functions of composition.

6. Free convection in an isothermal system

Although forced convection is undoubtedly the most common mode of mass exchange under practical conditions, there are nevertheless many instances where the transfer of matter by natural convection is the predominating factor. The present analysis will therefore conclude with a discussion of mass transfer by free convection, under laminar boundary-layer flow conditions, in the case where $B \gg 1$.

We note first of all that in natural convection the fluid motion is induced by the buoyancy forces, which are in turn generated by a variable density field. Density variations, on the other hand, may be caused by either temperature or composition gradients in the fluid, so that a general investigation of free-convection mass transfer will have to consider the simultaneous effect of momentum, mass and heat exchange between a stationary surface and the surrounding medium. In the interest of simplicity, however, we shall limit ourselves to an isothermal system with constant fluid properties-apart from the density in the buoyancy term-because, as can be verified without much difficulty, the results to be derived below may be generalized in the manner already shown for the forced convection case.

Under these restrictions, then, we have that (Schlichting 1960; Acrivos 1960c)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \quad (55)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = G(x)\,\theta+\frac{\partial^2 u}{\partial y^2}, \quad (56)$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y} = \frac{1}{Sc}\frac{\partial^2 \theta}{\partial y^2}, \quad (57)$$

where, again, both dimensionless and stretched laminar boundary-layer coordinates have been used. These are defined exactly as in forced convection, except that

$$U_\infty \equiv \{Lg\beta(W_s - W_\infty)\}^{\frac{1}{2}}, \tag{58}$$

where g is the gravitational acceleration and β is the expansion coefficient to be obtained from

$$\rho_\infty/\rho = 1+\beta\theta. \tag{59}$$

In addition, $G(x) \equiv \sin\epsilon$, where ϵ is the angle between the normal to the surface and the force of gravity.

It is quite clear now that, if we formally set

$$G(x) \equiv U\,dU/dx, \tag{60}$$

equations (55) to (57) become surprisingly similar to the forced convection equations (1) to (3). Therefore, except for a few minor modifications, the solutions of the previous sections can be made to apply to this free-convection problem.

(a) *Solution for small or moderate Sc*

Equations (55) to (60) may be rearranged once again by using the transformations given by equations (16) and (17), and by assuming that Λ is relatively independent of ξ. This leads, as a first approximation, to

$$f''' + ff'' + \Lambda(\theta - f'^2) = 0, \tag{61a}$$

$$\theta'' + Scf\theta' = 0, \tag{61b}$$

with the boundary conditions:

$$\left.\begin{aligned} &\text{at } \eta = 0, \quad \theta = 1, \quad f' = 0, \quad f \equiv -\alpha = (B/Sc)\,\theta'; \\ &\text{at } \eta = \infty, \quad \theta = 0, \quad f' = 0. \end{aligned}\right\} \tag{62}$$

This is a system of equation almost identical to equations (18) and (19), for which up to now only a few solutions have been made available with $B \neq 0$ (Acrivos 1960*a*; Eichhorn 1960). The $B \to \infty$ asymptote may, on the other hand, be obtained readily by simply following the steps which were outlined in § 5. In other words, since $\theta = 1$ in the 'inner' region, the 'inner' perturbation expansion for equation (61*a*) is again given by an expression similar to equations (25) and (47):

$$1 - f'^2 = \lambda^{2\Lambda} + \frac{2\Lambda(2\Lambda-1)\,\lambda^{2\Lambda}}{\alpha^2}\int_\lambda^1 \frac{(1-x^{2\Lambda})^{\frac{1}{2}}}{x^3}\,dx,$$

where $\lambda \equiv -f/\alpha$, while, in the 'outer' or $f \sim 0$ region,

$$\frac{d^3f}{d\eta_1^3} + f\frac{d^2f}{d\eta_1^2} + \Lambda\left[\theta - \left(\frac{df}{d\eta_1}\right)^2\right] = 0, \quad \frac{d^2\theta}{d\eta_1^2} + Scf\frac{d\theta}{d\eta_1} = 0, \tag{63}$$

with boundary conditions:

$$\left.\begin{aligned} &f' = 1, \quad \theta = 1 \quad \text{at} \quad \eta_1 = -\infty; \\ &f = 0 \quad \text{at} \quad \eta_1 = 0; \qquad f' = \theta = 0 \quad \text{at} \quad \eta_1 = \infty. \end{aligned}\right\} \tag{64}$$

Therefore, by comparing our results with those of the previous section we can immediately conclude that, in the limit $B \to \infty$:

$$j_1 \equiv \frac{jL}{\rho D}\left(\frac{\nu}{U_\infty L}\right)^{\frac{1}{2}} \left\{\frac{\left(2\int_0^x U\,dx\right)^{\frac{1}{2}}}{U(x)}\right\} \left(\frac{\sqrt{\pi}\,\Gamma(1+\Lambda^{-1})}{2Sc\Gamma(\frac{1}{2}+\Lambda^{-1})}\right)^{\frac{1}{2}} = \alpha_1 = B_1 \exp(-\alpha_1^2), \tag{65}$$

where
$$B_1 \equiv -\left(\frac{d\theta}{d\eta_1}\right)_{\eta_1=0}\left(\frac{2\pi}{Sc}\right)^{\frac{1}{2}} B\left(\frac{\Gamma(1+\Lambda^{-1})}{4\sqrt{\pi}\,\Gamma(\frac{1}{2}+\Lambda^{-1})}\right)^{\frac{1}{2}}$$
$$\times \exp\left\{-Sc\frac{(\Lambda-\frac{1}{2})}{(2-\Lambda)}[\Psi(1+\Lambda^{-1})-0{\cdot}03648]-ScE\right\} \tag{66}$$

and
$$E \equiv -\int_{-\infty}^{0}(1-f')f\,d\eta_1, \tag{67}$$

and where both E and $(d\theta/d\eta_1)_{\eta_1=0}$ must in general be obtained numerically from equations (63) and (64).

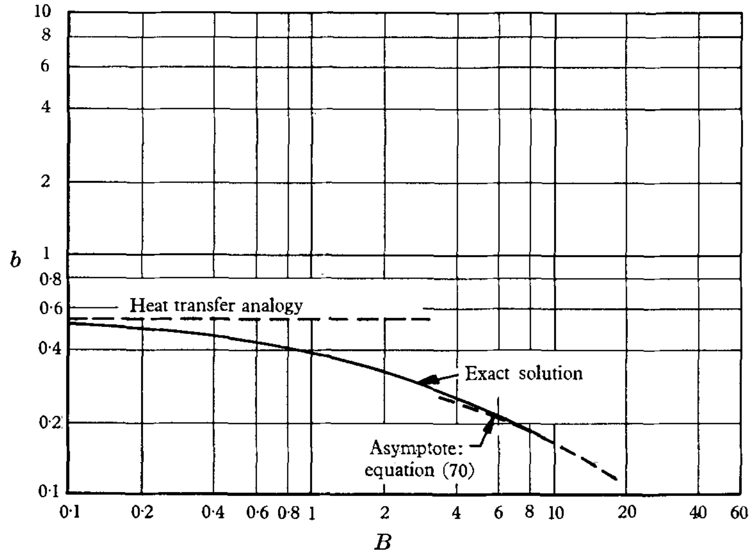

FIGURE 6. Free convection for large Sc; b is defined by equation (69).

(b) *Solution for* $Sc \gg 1$

It remains now to present the solution for large Schmidt numbers which fortunately can be made to hold for surfaces of arbitrary geometries. This is so because the inertia terms of the equation of motion may be neglected in the limit $Sc \to \infty$ (Morgan & Warner 1956), which in turn makes it possible to apply a similarity transformation to the free-convection equations and reduce them to a pair of ordinary differential equations (Acrivos 1960*c*). Thus, if

$$u = G^{\frac{1}{3}}\left(\frac{4}{3Sc}\int_0^x G^{\frac{1}{3}}dx\right)^{\frac{1}{2}} f'(\eta), \quad \text{where} \quad \eta \equiv yG^{\frac{1}{3}}\left[3Sc\Big/4\int_0^x G^{\frac{1}{3}}dx\right]^{\frac{1}{4}}$$

and if the inertia terms of equation (56) are omitted, then

$$\left.\begin{aligned} &f''' + \theta = 0, \quad \theta'' + f\theta' = 0 \\ &\text{with boundary conditions:} \\ &f'(0) = 0, \quad \theta(0) = 1, \quad f(0) = B\theta'(0), \quad f''(\infty) = 0, \quad \theta(\infty) = 0. \end{aligned}\right\} \tag{68}$$

It follows, therefore, that the quantity

$$\frac{jL}{\rho D}\left(\frac{\nu}{U_\infty L}\right)^{\frac{1}{2}}\frac{1}{BG^{\frac{1}{3}}}\left[4\int_0^x G^{\frac{1}{2}}dx\Big/3Sc\right]^{\frac{1}{3}} = -\theta'(0) \equiv b \tag{69}$$

is a unique function of B which, as can be shown in a straightforward manner, has as its two asymptotes:

$$b = 0{\cdot}540 \quad \text{for} \quad B \to 0$$

and

$$b = 0{\cdot}407(Bb)^{\frac{1}{3}}\exp\{-0{\cdot}541(Bb)^{\frac{2}{3}}\} \quad \text{for} \quad B \gg 1. \tag{70}$$

Again, as can be seen from figure 6, these two expressions describe rather accurately the behaviour of the function b, and that, as was the case with the previous examples of this analysis, the asymptotic expression may be used to represent b even when B is as low as 3.

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Appendix

We wish to evaluate

$$j \equiv \int_0^1 \frac{\lambda^{2\Lambda+1}}{(1-\lambda^{2\Lambda})^{\frac{2}{3}}}\left(\int_\lambda^1 \frac{(1-x^{2\Lambda})^{\frac{1}{2}}}{x^3}\,dx\right)d\lambda.$$

Let $z = \lambda^{2\Lambda}$ so that

$$j = \frac{1}{(2\Lambda)^2}\int_0^1 \frac{z^{1/\Lambda}}{(1-z)^{\frac{2}{3}}}\left(\int_z^1 \frac{(1-y)^{\frac{1}{2}}}{y^{1+1/\Lambda}}\,dy\right)dz.$$

However [see Morse & Feshbach 1953, p. 592, eq. (5.3.19)],

$$\int_z^1 \frac{(1-y)^{\frac{1}{2}}}{y^{1+1/\Lambda}}\,dy = \int_0^{1-z} \frac{y^{\frac{1}{2}}}{(1-y)^{1+1/\Lambda}}\,dy = \tfrac{2}{3}(1-z)^{\frac{3}{2}}F(1+\Lambda^{-1}, \tfrac{3}{2}, \tfrac{5}{2}, 1-z),$$

where F is the hypergeometric function. On the other hand (see Erdélyi *et al.* 1953, p. 105),

$$F(a,b,c,z) = (1-z)^{c-a-b}F(c-a, c-b, c, z)$$

and therefore

$$j = \frac{2}{3(2\Lambda)^2}\int_0^1 F(\tfrac{3}{2}-\Lambda^{-1}, 1, \tfrac{5}{2}, z)\,dz.$$

This may be simplified still further by recalling that (Erdélyi *et al.* 1953, p. 116)

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^\infty e^{-bt}(1-e^{-t})^{c-b-1}(1-z\,e^{-t})\,dt,$$

which means that, since

$$\int_0^1 (1-z\,e^{-t})^{-a}\,dz = \frac{e^t}{1-a}[1-(1-e^{-t})^{1-a}],$$

$$j = \frac{1}{(2\Lambda)(2-\Lambda)}\int_0^\infty \{(1-e^{-t})^{\frac{1}{2}} - (1-e^{-t})^{1/\Lambda}\}\,dt.$$

This last integral may finally be determined as follows. We note from a table of integral transforms (Erdélyi *et al.* 1954, p. 14) that

$$\int_0^\infty e^{-pt}(1-e^{-t})^{\nu-1}\,dt = \frac{\Gamma(\nu)\,\Gamma(p)}{\Gamma(\nu+p)},$$

from which we readily obtain

$$\begin{aligned} j &= \frac{1}{(2\Lambda)(2-\Lambda)}\lim_{p\to 0}\left[\frac{\Gamma(\frac{3}{2})\,\Gamma(p)}{\Gamma(\frac{3}{2}+p)} - \frac{\Gamma(1+\Lambda^{-1})\,\Gamma(p)}{\Gamma(1+\Lambda^{-1}+p)}\right] \\ &= \frac{1}{(2\Lambda)(2-\Lambda)}[\Psi(1+\Lambda^{-1}) - \Psi(\tfrac{3}{2})], \end{aligned}$$

where $$\Psi(x) \equiv d(\ln\Gamma)/dx.$$

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